

Pure elastic bending of a prismatic bar

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units, with the exception of the force that is expressed in $\text{N} \times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: $\underline{x}$ of which: $\underline{x} \equiv \{x_i; i=1,3\}$ $[x_i] = [\text{length}]$ $\mathcal{R}(\underline{x}_i) \equiv (-\infty, \infty)$, $\underline{x}$ a point of the deformed medium.

Coordinate versors: $\{\mathbf{v}_i; i=1,3\}$

Unknown functions: $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ of which: $\mathbf{s}_i = \mathbf{x}_i - \mathbf{X}_i$, $[\mathbf{s}_i] = [\text{length}]$, $\underline{X} \equiv \{X_i; i=1,3\}$, $\underline{X}$ the position of the point $\underline{x}$ in the undeformed medium, $\mathbf{s} \equiv \sum_{i=1,3} (\mathbf{s}_i \cdot \mathbf{v}_i)$, $\mathbf{s}$ the displacement of the point $\underline{X}$, $\{\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}] = [\text{stress}]$, $\tau_{ij} = \tau_{ji}$.

Differential analytical model:

$$\partial \tau_{11}(\underline{x}) / \partial x_1 + \partial \tau_{12}(\underline{x}) / \partial x_2 + \partial \tau_{13}(\underline{x}) / \partial x_3 + F_1(\underline{x}) = 0$$

$$\partial \tau_{12}(\underline{x}) / \partial x_1 + \partial \tau_{22}(\underline{x}) / \partial x_2 + \partial \tau_{23}(\underline{x}) / \partial x_3 + F_2(\underline{x}) = 0$$

$$\partial \tau_{13}(\underline{x}) / \partial x_1 + \partial \tau_{23}(\underline{x}) / \partial x_2 + \partial \tau_{33}(\underline{x}) / \partial x_3 + F_3(\underline{x}) = 0$$

$$\{(1+\nu) \cdot \tau_{ij}(\underline{x}) - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) - E \cdot (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = 0; j=i,3; i=1,3\}$$

of which: $\mathbf{F} \equiv \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, $\mathbf{F}$ the body force per unit volume, $\{\delta_{ij}=0; \forall i \neq j\}$ $\{\delta_{ij}=1; \forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$$\varepsilon_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = (1+\nu) \cdot \tau_{ij}(\underline{x}) / E - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) / E \quad (1)$$

$$\omega_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j - \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 \quad (2)$$

$$\mathbf{T}_i(\underline{x}) = \sum_{j=1,3} (\tau_{ji}(\underline{x}) \cdot \mathbf{n}_j(\underline{x})) \quad (3)$$

$$\mathbf{s}_i(\underline{x}_B) = \mathbf{s}_i(\underline{x}_A) + \sum_{j=1,3} (\omega_{ij}(\underline{x}_A) \cdot (\mathbf{x}_{Bj} - \mathbf{x}_{Aj})) + \int_{A,B} (\Theta_i(\mathbf{c}) \cdot d\mathbf{c}) \quad (4)$$

$$\Theta_i(\mathbf{c}) \equiv \sum_{j=1,3} (\varepsilon_{ij}(\underline{x}(\mathbf{c})) \cdot \mathbf{x}_j'(\mathbf{c}) + (\mathbf{x}_{Bj} - \mathbf{x}_j(\mathbf{c})) \cdot \sum_{k=1,3} ((\partial \varepsilon_{ik}(\underline{x}(\mathbf{c})) / \partial x_j - \partial \varepsilon_{jk}(\underline{x}(\mathbf{c})) / \partial x_i) \cdot \mathbf{x}_k'(\mathbf{c}))) \quad (5)$$

of which [here](#), $\varepsilon_{ij} = \varepsilon_{ji}$, $\mathbf{T}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{T}_i(\underline{x}) \cdot \mathbf{v}_i)$ $\mathbf{n}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{n}_i(\underline{x}) \cdot \mathbf{v}_i)$, $\mathbf{T}(\underline{x})$ the stress vector in a point of a plane with normal outward versor $\mathbf{n}(\underline{x})$, $\underline{x}(\mathbf{c}) \equiv \{x_i(\mathbf{c}); i=1,3\}$ $\underline{x}_A \equiv \{x_{Ai}; i=1,3\} = \underline{x}(A)$ $\underline{x}_B \equiv \{x_{Bi}; i=1,3\} = \underline{x}(B)$

Definition set: $\{\underline{x}/0 \leq x_1 \leq L_1; 0 \leq x_2 \leq L_2; 0 \leq x_3 \leq L_3\}$ $L_1=1$ $L_2=10$ $L_3=10$.

Conditions:

$$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0 \quad \{\varsigma_i(\underline{x}_A)=0; i=1,3\} \quad \partial \varsigma_1(\underline{x}_A)/\partial x_2=\partial \varsigma_1(\underline{x}_A)/\partial x_3=\partial \varsigma_2(\underline{x}_A)/\partial x_3=0 \quad (6)$$

where $\underline{x}_A \equiv \{L_1/2, 0, L_3/2\}$.

$$\begin{aligned} T_1(0, x_2, x_3) &= T_2(0, x_2, x_3) = T_3(0, x_2, x_3) = T_1(L_1, x_2, x_3) = T_2(L_1, x_2, x_3) = T_3(L_1, x_2, x_3) = T_1(x_1, x_2, 0) = \\ T_2(x_1, x_2, 0) &= T_3(x_1, x_2, 0) = T_1(x_1, x_2, L_3) = T_2(x_1, x_2, L_3) = T_3(x_1, x_2, L_3) = T_1(x_1, 0, x_3) = T_3(x_1, 0, x_3) = \\ T_1(x_1, L_2, x_3) &= T_3(x_1, L_2, x_3) = 0 \quad T_2(x_1, 0, x_3) = -P \cdot ((2/L_3) \cdot x_3 - 1) \quad T_2(x_1, L_2, x_3) = P \cdot ((2/L_3) \cdot x_3 - 1) \quad P=0.1 \end{aligned}$$

From these and (3) follows

$$\begin{aligned} \tau_{11}(0, x_2, x_3) &= \tau_{12}(0, x_2, x_3) = \tau_{13}(0, x_2, x_3) = \tau_{11}(L_1, x_2, x_3) = \tau_{12}(L_1, x_2, x_3) = \tau_{13}(L_1, x_2, x_3) = \tau_{13}(x_1, x_2, 0) = \\ \tau_{23}(x_1, x_2, 0) &= \tau_{33}(x_1, x_2, 0) = \tau_{13}(x_1, x_2, L_3) = \tau_{23}(x_1, x_2, L_3) = \tau_{33}(x_1, x_2, L_3) = \tau_{12}(x_1, 0, x_3) = \tau_{23}(x_1, 0, x_3) = \\ \tau_{12}(x_1, L_2, x_3) &= \tau_{23}(x_1, L_2, x_3) = 0 \quad \tau_{22}(x_1, 0, x_3) = \tau_{22}(x_1, L_2, x_3) = P \cdot ((2/L_3) \cdot x_3 - 1) \end{aligned}$$

Related files: [mad.txt](#)

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x})=\tau_{12}(\underline{x})=\tau_{13}(\underline{x})=\tau_{33}(\underline{x})=\tau_{23}(\underline{x})=0$ $\tau_{22}(\underline{x})=P \cdot ((2/L_3) \cdot x_3 - 1)$. These and (1) imply

$$\varepsilon_{12}(\underline{x})=\varepsilon_{13}(\underline{x})=\varepsilon_{23}(\underline{x})=0 \quad \varepsilon_{11}(\underline{x})=\varepsilon_{33}(\underline{x})=-v \cdot P \cdot ((2/L_3) \cdot x_3 - 1)/E \quad \varepsilon_{22}(\underline{x})=P \cdot ((2/L_3) \cdot x_3 - 1)/E \quad (7)$$

From these, (1), (2), $\partial \varsigma_i(\underline{x})/\partial x_j = \varepsilon_{ij}(\underline{x}) + \omega_{ij}(\underline{x})$ and (6) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{\varsigma_i(\underline{x}_A)=0; i=1,3\}$ and (4) imply

$$\varsigma_i(\underline{x}_B) = \int_{A,B} (\Theta_i(c) \cdot dc) \quad (8)$$

of which $\underline{x}(A) \equiv \underline{x}_A$.

Are placed

$$\begin{aligned} \int_{A,B} (\Theta_i(c) \cdot dc) &= \int_{A,P} (\Theta_i(c) \cdot dc) + \int_{P,Q} (\Theta_i(c) \cdot dc) + \int_{Q,B} (\Theta_i(c) \cdot dc) \quad \underline{x}(P) \equiv \{L_1/2, x_2, L_3/2\} \quad \underline{x}(Q) \equiv \{x_1, x_2, L_3/2\} \\ \underline{x}(B) &\equiv \{x_1, x_2, x_3\} \quad \{x_1'(c)=x_3'(c)=0, x_2'(c)=1; \forall c \in [A, P]\} \quad \{x_2'(c)=x_3'(c)=0, x_1'(c)=1; \forall c \in [P, Q]\} \\ \{x_1'(c)=x_2'(c)=0, x_3'(c)=1; \forall c \in [Q, B]\} & \end{aligned} \quad (9)$$

These, (5) and (7) imply

$$\begin{aligned} \{\Theta_1(c)=0, \Theta_2(c)=P \cdot ((2/L_3) \cdot x_{B3} - 1)/E, \Theta_3(c)=-P \cdot (2/L_3) \cdot (x_{B2} - x_2(c))/E; \forall c \in [A, P]\} \\ \{\Theta_1(c)=-v \cdot P \cdot (x_{B3} \cdot (2/L_3) - 1)/E, \Theta_2(c)=0, \Theta_3(c)=v \cdot P \cdot (2/L_3) \cdot (x_{B1} - x_1(c))/E; \forall c \in [P, Q]\} \\ \{\Theta_1(c)=0, \Theta_2(c)=0, \Theta_3(c)=-v \cdot P \cdot ((2/L_3) \cdot x_3(c) - 1)/E; \forall c \in [Q, B]\} \end{aligned}$$

From these, (8) and (9) follows

$$\begin{aligned} \varsigma_1(\underline{x}) &= -v \cdot P \cdot (x_3 \cdot (2/L_3) - 1) \cdot (x_1 - L_1/2)/E \quad \varsigma_2(\underline{x}) = P \cdot ((2/L_3) \cdot x_3 - 1) \cdot x_2/E \\ \varsigma_3(\underline{x}) &= P \cdot (v \cdot (x_1 - L_1/2)^2 - x_2^2 - v \cdot (L_3^2/4 + x_3^2 - L_3 \cdot x_3))/(L_3 \cdot E) \end{aligned} \quad (10)$$

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to $\underline{x}$, $\underline{X}$ determined by means of $X_i = x_i - \varpi_i$ and (10), and $\underline{X}$ determined by means of $X_i = x_i - \varpi_i$ where ϖ_i is calculated by PEEI.

Case 3-3-3: [points-3-3-3.txt](#), [mem-3-3-3.bin](#), [cond-3-3-3.txt](#), [sol-3-3-3.txt](#), [plot-3-3-3-1.jpg](#), [plot-3-3-3-2.jpg](#), [plot-3-3-3-3.jpg](#)

Case 3-5-3: [points-3-5-3.txt](#), [mem-3-5-3.bin](#), [cond-3-5-3.txt](#), [sol-3-5-3.txt](#), [plot-3-5-3-1.jpg](#), [plot-3-5-3-2.jpg](#), [plot-3-5-3-3.jpg](#)

Case 3-7-3: [points-3-7-3.txt](#), [mem-3-7-3.bin](#), [cond-3-7-3.txt](#), [sol-3-7-3.txt](#), [plot-3-7-3-1.jpg](#), [plot-3-7-3-2.jpg](#), [plot-3-7-3-3.jpg](#)

Case 3-9-3: [points-3-9-3.txt](#), [mem-3-9-3.bin](#), [cond-3-9-3.txt](#), [sol-3-9-3.txt](#), [plot-3-9-3-1.jpg](#), [plot-3-9-3-2.jpg](#), [plot-3-9-3-3.jpg](#)

Case 5-3-5: [points-5-3-5.txt](#), [mem-5-3-5.bin](#), [cond-5-3-5.txt](#), [sol-5-3-5.txt](#), [plot-5-3-5-1.jpg](#), [plot-5-3-5-2.jpg](#), [plot-5-3-5-3.jpg](#)

Case 5-5-5: [points-5-5-5.txt](#), [mem-5-5-5.bin](#), [cond-5-5-5.txt](#), [sol-5-5-5.txt](#), [plot-5-5-5-1.jpg](#), [plot-5-5-5-2.jpg](#), [plot-5-5-5-3.jpg](#)

Case 5-7-5: [points-5-7-5.txt](#), [mem-5-7-5.bin](#), [cond-5-7-5.txt](#), [sol-5-7-5.txt](#), [plot-5-7-5-1.jpg](#), [plot-5-7-5-2.jpg](#), [plot-5-7-5-3.jpg](#)

Case 5-9-5: [points-5-9-5.txt](#), [mem-5-9-5.bin](#), [cond-5-9-5.txt](#), [sol-5-9-5.txt](#), [plot-5-9-5-1.jpg](#), [plot-5-9-5-2.jpg](#), [plot-5-9-5-3.jpg](#)

Case 7-3-7: [points-7-3-7.txt](#), [mem-7-3-7.bin](#), [cond-7-3-7.txt](#), [sol-7-3-7.txt](#), [plot-7-3-7-1.jpg](#), [plot-7-3-7-2.jpg](#), [plot-7-3-7-3.jpg](#)

Case 7-5-7: [points-7-5-7.txt](#), [mem-7-5-7.bin](#), [cond-7-5-7.txt](#), [sol-7-5-7.txt](#), [plot-7-5-7-1.jpg](#), [plot-7-5-7-2.jpg](#), [plot-7-5-7-3.jpg](#)

Case 7-7-7: [points-7-7-7.txt](#), [mem-7-7-7.bin](#), [cond-7-7-7.txt](#), [sol-7-7-7.txt](#), [plot-7-7-7-1.jpg](#), [plot-7-7-7-2.jpg](#), [plot-7-7-7-3.jpg](#)

Case 7-9-7: [points-7-9-7.txt](#), [mem-7-9-7.bin](#), [cond-7-9-7.txt](#), [sol-7-9-7.txt](#), [plot-7-9-7-1.jpg](#), [plot-7-9-7-2.jpg](#), [plot-7-9-7-3.jpg](#)

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow